

# Selftuned Massive Spin-2

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## Abstract

We calculate the cubic order terms in a covariant theory that gives a nonlinear completion of the Fierz-Pauli massive spin-2 action. The resulting terms have specially tuned coefficients guarantying the absence of a ghost at this order in the decoupling limit. We show in this limit that: (1) The quadratic theory propagates helicity-2, 1, and helicity-0 states of massive spin-2. (2) The cubic terms with six derivatives – which would give ghosts on local backgrounds – cancel out automatically. (3) There is a four-derivative cubic term for the helicity-0 field, that has been known to be ghost-free on any local background. (4) There are four-derivative cubic terms that mix two helicity-0 fields with one helicity-2, or two helicity-1 fields with one helicity-0; none of them give ghosts on local backgrounds. (5) In the absence of external sources, all the cubic mixing terms can be removed by nonlinear redefinitions of the helicity-2 and helicity-1 fields. Notably, the helicity-2 redefinition generates the quartic Galileon term. These findings hint to an underlying nonlinearly realized symmetry, that should be responsible for what appears as the accidental cancellation of the ghost.

# 1. Introduction and summary

An effective field theory for massive spin-2 is motivated by spin-2 QCD resonances (glueballs, quark-antiquark mesons, or their mixtures) which become long-lived in the limit of a large number of colors. It is also motivated by massive gravity, and a possibility of having dark energy made of “condensate” of massive gravitons. In what follows we will be discussing a classical theory of massive gravity on asymptotically flat space-time.

The Fierz-Pauli (FP) Lagrangian [1] is a linear theory describing a massive spin-2 state on flat space, without any ghosts or tachyons [2]. Recently, a non-linear completion to the FP theory to all orders was proposed in Refs. [3, 4]. The gravitational field in this approach is described by an extended metric tensor  $\tilde{g}_{\mu\nu}(x, u)$ , with  $\mu, \nu = 0, 1, 2, 3$ , which is labeled by a continuous dimensionless parameter  $u$ . The matter fields do not depend on  $u$ , but couple to the metric tensor  $g_{\mu\nu}(x) \equiv \tilde{g}_{\mu\nu}(x, u = 0)$ . The Lagrangian density for the gravitational field alone reads as follows:

$$\mathcal{L} = M_{\text{Pl}}^2 \sqrt{g} R - \frac{M_{\text{Pl}}^2 m^2}{2} \int_{-1}^{+1} du \sqrt{\tilde{g}} (k_{\mu\nu}^2 - k^2), \quad (1)$$

where  $R$  is the Ricci scalar of the metric  $g_{\mu\nu}(x)$ , while  $k_{\mu\nu} \equiv \frac{1}{2} \partial_u \tilde{g}_{\mu\nu}$ ,  $k \equiv \tilde{g}^{\mu\nu} k_{\mu\nu}$ ; all indices in the Einstein-Hilbert term in (1) are raised by  $g^{\mu\nu}$ , while those in the second term by  $\tilde{g}^{\mu\nu}$ . The  $\mathbf{Z}_2$  symmetry is imposed on the fields,  $\tilde{g}_{\mu\nu}(x, u) = \tilde{g}_{\mu\nu}(x, -u)$ . The “ $u$ -dimension” is not dynamical since fields have no ordinary derivative terms for  $u > 0$ , and there are no  $g_{uu}$  or  $g_{u\mu}$  components of the metric to vary.

Massive gravity is obtained by requiring that the space-time geometry at  $u = 1$  is flat. One way to impose this boundary condition is to require that  $\tilde{g}_{\mu\nu}(x, u = 1) = \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1)$  [3, 4]. Then, the auxiliary dimension can be “integrated out”. This produces quadratic and nonlinear terms in the resulting effective Lagrangian with specially tuned coefficients. In the quadratic order one gets the FP term [3, 4]. We will show below that the cubic order result reads as follows:

$$\mathcal{L} = M_{\text{Pl}}^2 \sqrt{g} R - \frac{M_{\text{Pl}}^2 m^2}{4} \left( h_{\mu\nu}^2 - h^2 - h_\mu^\nu h_\nu^\alpha h_\alpha^\mu + \frac{5}{4} h h_{\alpha\beta}^2 - \frac{1}{4} h^3 \right), \quad (2)$$

where  $h_{\mu\nu} \equiv g_{\mu\nu} - \eta_{\mu\nu}$ ; all the indices in the mass term in (2) are raised by  $\eta^{\mu\nu}$ , and the  $\sqrt{g}R$  term in (2) should also be expanded up to the cubic order.

For small perturbations the quadratic FP term in (2) describes five degrees of freedom of massive spin-2. Naively, this would seem enough for perturbative consistency. Regrettably, the issue is more subtle: to see a potential problem one can consider a weak field of a locally-nontrivial asymptotically-flat solution – say a weak field of a static lump of matter (below we will refer such backgrounds as *local*). Then, the expression for  $h_{\mu\nu}$  can be decomposed as

$$h_{\mu\nu} = h_{\mu\nu}^{cl} + \delta_{\mu\nu}, \quad (3)$$

where  $h_{\mu\nu}^{cl}$  denotes the weak field of the local background and  $\delta_{\mu\nu}$  denotes small fluctuations about it,  $1 \gg h_{\mu\nu}^{cl} \gg \delta_{\mu\nu}$ . Substituting (3) into (2) one gets new quadratic terms for the fluctuations,  $h^{cl}\delta\delta$ . Since  $h^{cl} \ll 1$  these new terms are smaller than the  $\delta\delta$  terms. Nevertheless, they may destroy the delicate balance of the FP term and introduce a ghost. In a generic nonlinear completion of the FP massive gravity such a ghost does appear as a sixth degree of freedom [5].

In order for the sixth mode not to emerge a very special tuning of the coefficients of the quadratic and cubic terms is needed, as was observed in [6] and worked out in [7]. What is interesting, is that the theory (1) automatically produces such tuned coefficients up to the cubic order!

There could have been four arbitrary coefficients in front of the five quadratic and cubic order terms in (2). In the initial Lagrangian (1) we tuned the coefficient between the two terms under the  $u$ -integral. As a result, we have automatically generated four “good” coefficients in (2)<sup>1</sup>. This hints towards a hidden symmetry of (1) which may be responsible for such arrangements.

One goal of the present paper is to derive the terms presented in (2) from the Lagrangian (1). Furthermore, after observing the cancellation of the “ghost terms”, we will study the remaining nonlinear interactions of the helicity  $\pm 2$ ,  $\pm 1$  and helicity-0 states.

Thus, we will be looking at the cubic order Lagrangian (2) in the decoupling limit where the nonlinear dynamics of all the helicities can be made manifest. This method – first used for massive non-Abelian gauge fields in Ref. [8], and developed for massive gravity in [6] – proved to be successful in identifying the presence or absence of the sixth degree of freedom in nonlinear theories [6, 9, 7, 10]. The limit we consider reads as follows:

$$m \rightarrow 0, \quad M_{\text{Pl}} \rightarrow \infty, \quad \Lambda_3 \equiv (m^2 M_{\text{Pl}})^{1/3} \text{ is fixed.} \quad (4)$$

By taking this limit in (2) we show that: (I) The quadratic theory propagates the helicity  $\pm 2$  modes described by the tensor field  $\bar{h}_{\mu\nu}$ , the helicity  $\pm 1$  modes described by the vector field  $A_\mu$ , and the helicity-0 mode described by the scalar field  $\pi$  – all with canonical kinetic terms. (II) The cubic terms with six derivatives,  $(\partial\partial\pi)^3$ , which would give rise to the sixth degree of freedom on local backgrounds, cancel out automatically due to the special values of the coefficients of the quadratic and cubic terms in (2). (III) There is a four-derivative cubic term for the helicity-0 mode,  $\square\pi(\partial\pi)^2$ , which was first found in the context of the DGP model [11] in Ref. [12]. This term is ghost-free for local backgrounds; it is also invariant under the “galilean” transformation in the  $\pi$  space,  $\partial_\mu\pi \rightarrow \partial_\mu\pi + v_\mu$ , with  $v_\mu$  being a constant four-vector. (IV) There are four-derivative cubic terms that mix two helicity-0 fields with one helicity-2, such as  $\bar{h}((\square\pi)^2 - (\partial\partial\pi)^2)$ , or two helicity-1 fields with one helicity-0, such

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<sup>1</sup>In general, there exists a one-parameter family of cubic order terms for which the sixth derivative terms for helicity-0 do cancel out,  $(h_{\mu\nu}^2 - h^2 + c_1 h_\mu^\nu h_\nu^\alpha h_\alpha^\mu - \frac{6c_1+1}{4} h h_{\alpha\beta}^2 + \frac{2c_1+1}{4} h^3)$ . The theory (1) with the boundary conditions used here gives  $c_1 = -1$ . From this point of view, (1) had to generate only three “good” coefficients up to this order.

as  $\partial\partial\pi(\partial A)^2$ . All of them are “galilean” invariant, and none of these terms gives rise to ghosts on the local backgrounds. (V) If the external sources are ignored (or outside of localized sources) all the cubic mixings between the helicity-2 and helicity-0, and helicity-1 and helicity-0, can be removed by a nonlinear redefinition of the helicity-2 and helicity-1 fields, respectively. What remains is the term,  $\Box\pi(\partial\pi)^2$ . Interestingly, the above redefinition of  $\bar{h}_{\mu\nu}$  generates also the quartic Galileon term,  $(\partial\pi)^2((\Box\pi)^2 - (\partial\partial\pi)^2)$  [13]. The latter is known to be ghost-free [13] (for more general studies of the Galileon, see, [14]).

That the obtained terms contain the four-derivative cubic term – the cubic Galileon  $\Box\pi(\partial\pi)^2$  – identical to the one found in Ref. [12] in the context of the DGP model, is a hint: The Galileon terms emerge in theories with spontaneously broken symmetries [15]. Hence, our findings suggest that there should be an underlying nonlinearly realized symmetry of (1), which is responsible for the cancellations of the “ghost terms”. It is conceivable that this symmetry is related to a spontaneously broken 5D reparametrization invariance hiddenly present in the model (1). This symmetry should also be helpful in addressing the issue of stability of (1) with respect to quantum gravity loops, which has been left open for now.

So far we have focused on the cubic order. How about an arbitrary  $n'$ th order terms,  $(h^{cl})^{n-2}\delta\delta$ ? Making sure that the sixth degree of freedom does not show up order-by-order would be a tedious program<sup>2</sup>. However, we are in a better position here as the Lagrangian (1) sums up all the polynomial terms in  $h_{\mu\nu}$ . Then, the presence of a ghost could be seen by calculating the exact Hamiltonian. Boulware and Deser (BD) [5] have shown that the sixth degree of freedom would in general appear in nonlinear massive gravity due to the loss of the Hamiltonian constraint: This leads to a Hamiltonian term that is proportional to positive powers of the canonical momenta, but is sign indefinite, hence, signaling the presence of a ghost.

Therefore, the absence of the BD term would be a good indicator that the sixth degree of freedom is not present<sup>3</sup>. The Hamiltonian for (1) was calculated in [3], where it was shown that the BD term cancels out. Hence, one should expect that the BD ghost does not appear in the order-by-order expansion of (1). Moreover, in Ref. [4] the decoupling limit of the theory was considered and it was shown to all orders that the leading terms arising at the scale  $\Lambda < \Lambda_3$ , that could give rise to ghosts, cancel out<sup>4</sup>.

There have been proposals in the literature to obtain the theory of massive spin-2 via a dynamical condensation mechanism (for recent works see, e.g., [17, 18] and

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<sup>2</sup>Since in each consecutive order these terms have smaller and smaller coefficients, one could try to show that after a certain order the ghost appears only above a certain UV cutoff of the low energy theory.

<sup>3</sup>It is not a guarantee, however, of the absence of other types of ghosts that may be present in a theory for some other reasons (e.g., introduced by hand). Here, we’re focusing on the ghost that may appear in massive gravity due to the nonlinear interactions.

<sup>4</sup>Ref. [7] concluded that no linear combination of the quartic order terms in  $h_{\mu\nu}$  can give a theory for  $\pi$  that would be ghost-free. This issue will be revisited in our forthcoming paper [16], with a different conclusion. Until then, we ignore the explicit quartic and higher terms in  $h_{\mu\nu}$ .

references therein). It would be interesting to see whether the cubic terms in these models can also automatically give rise to ghost-free structures for the  $\pi$  field.

## 2. Integrating out the auxiliary dimension

The goal here is to calculate order-by-order the  $u$ -dependence of the extended metric  $\tilde{g}_{\mu\nu}(x, u)$ , then substitute it back into (1), and integrate the latter w.r.t.  $u$ . This should give the effective Lagrangian written in terms of  $h_{\mu\nu}$  only. To fulfill this goal we introduce the notations

$$\tilde{g}_{\mu\nu}(x, u) = \eta_{\mu\nu} + H_{\mu\nu}^{(1)}(x, u) + H_{\mu\nu}^{(2)}(x, u) + H_{\mu\nu}^{(3)}(x, u) + \dots, \quad (5)$$

where  $H_{\mu\nu}^{(1)}(x, u)$ ,  $H_{\mu\nu}^{(2)}(x, u)$ ,  $\dots$  are perturbations in the corresponding order. Since  $\tilde{g}_{\mu\nu}(x, u=0) = g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x)$ , we require that

$$H_{\mu\nu}^{(1)}(x, u=0) = h_{\mu\nu}(x), \quad H_{\mu\nu}^{(2)}(x, u=0) = H_{\mu\nu}^{(3)}(x, u=0) = 0. \quad (6)$$

Furthermore, following Refs. [3, 4] we impose the boundary condition on the extended metric  $\tilde{g}_{\mu\nu}(x, u=1) = \eta_{\mu\nu}$ , which guarantees that the theory reduces to FP massive gravity in the quadratic approximation. This boundary condition implies that  $H_{\mu\nu}^{(1)}(x, u=1) = H_{\mu\nu}^{(2)}(x, u=1) = H_{\mu\nu}^{(3)}(x, u=1) = 0$ . Having set these, the calculation of the  $u$ -dependence of  $\tilde{g}_{\mu\nu}(x, u)$  is well defined.

The expression for the mass term in (2) in terms of  $H_{\mu\nu}^{(1)}(x, u)$ ,  $H_{\mu\nu}^{(2)}(x, u)$ ,  $\dots$ , up to cubic order, is straightforward to obtain:

$$\begin{aligned} -\frac{M_{\text{Pl}}^2 m^2}{2} 2 \int_0^{+1} \frac{du}{4} & \left( (\partial_u H_{\mu\nu}^{(1)})^2 + 2\partial_u H_{\mu\nu}^{(1)} \partial_u H_{\mu\nu}^{(2)} - 2H_{\mu\nu}^{(1)} \partial_u H_{\nu\alpha}^{(1)} \partial_u H_{\alpha\mu}^{(1)} - (\partial_u H^{(1)})^2 \right. \\ & \left. - 2\partial_u H^{(1)} \partial_u H^{(2)} + 2\partial_u H^{(1)} H_{\mu\nu}^{(1)} \partial_u H_{\mu\nu}^{(1)} + \frac{1}{2} H^{(1)} (\partial_u H_{\mu\nu}^{(1)})^2 - \frac{1}{2} H^{(1)} (\partial_u H^{(1)})^2 \right), \quad (7) \end{aligned}$$

where we used the fact that the metric is a  $\mathbf{Z}_2$  symmetric function of  $u$ , and set the integration limits from 0 to 1. Also, in (7) and in what follows, we use simplified notations with all lower-case indices contracted by the flat space-time metric.

To find the  $u$ -dependence of  $H_{\mu\nu}^{(1)}(x, u)$ ,  $H_{\mu\nu}^{(2)}(x, u)$ ,  $\dots$  we integrate the equations of motion. For this we vary the action (1) w.r.t.  $\tilde{g}_{\mu\nu}(x, u)$ . The resulting equations for  $u=0^+$  and  $0 < u \leq 1$  read respectively as follows:

$$G_{\mu\nu} - m^2 (k_{\mu\nu} - g_{\mu\nu} k) = T_{\mu\nu} / (2M_{\text{Pl}}^2), \quad (8)$$

and

$$\partial_u \left[ \sqrt{\tilde{g}} (k \tilde{g}^{\mu\nu} - k^{\mu\nu}) \right] = \frac{1}{2} \tilde{g}^{\mu\nu} \sqrt{\tilde{g}} (k^2 - k_{\alpha\beta}^2) + 2\sqrt{\tilde{g}} (k^{\mu\rho} k_{\rho}^{\nu} - k^{\mu\nu} k). \quad (9)$$

The latter is the equation that determines the “ $u$ -evolution” of the extended metric. With the boundary conditions specified above it is straightforward to solve (9) for

$H_{\mu\nu}^{(1)}(x, u)$ ,  $H_{\mu\nu}^{(2)}(x, u)$ , .... It turns out that only the solution for  $H_{\mu\nu}^{(1)}(x, u)$  is needed at the cubic order. The latter reads:

$$H_{\mu\nu}^{(1)}(x, u) = (1 - u)h_{\mu\nu}(x). \quad (10)$$

This expression for  $H_{\mu\nu}^{(1)}(x, u)$  guarantees that the two terms in (7), which contain the function  $H_{\mu\nu}^{(2)}(x, u)$ , integrate to zero due to the boundary conditions. Substituting the expression (10) into (1) and integrating it w.r.t.  $u$  we obtain

$$-\frac{M_{\text{Pl}}^2 m^2}{4} \left( h_{\mu\nu}^2 - h^2 - h_\mu^\nu h_\nu^\alpha h_\alpha^\mu + \frac{5}{4} h h_{\alpha\beta}^2 - \frac{1}{4} h^3 \right), \quad (11)$$

which is the mass term presented earlier in (2).

### 3. Extracting the longitudinal mode

We begin by rewriting the Lagrangian density (2) in a manifestly covariant form. For this, following Ref. [6], we introduce a covariant tensor field  $H_{\mu\nu}$ , which is related to  $g_{\mu\nu}$  as follows:

$$H_{\mu\nu} = g_{\mu\nu} - \eta_{ab} \frac{\partial \phi^a(x)}{\partial x^\mu} \frac{\partial \phi^b(x)}{\partial x^\nu}, \quad (12)$$

where  $\phi^a(x)$  are just four scalars;  $a, b = 0, 1, 2, 3$ , and  $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$  is the flat metric on the field space of the scalars. This construction guarantees that  $H_{\mu\nu}$  transforms as a covariant symmetric rank-2 tensor.

Furthermore, it is convenient to decompose the scalars as  $\phi^a(x) = x^a - \pi^a(x)$ , where  $x^a \equiv \delta_\mu^a x^\mu$ . This decomposition specifies that under the general coordinate transformations,  $x^\mu \rightarrow x^\mu + \zeta^\mu(x)$ , the fields  $\pi^a(x)$  transform as  $\pi^a \rightarrow \pi^a + \delta_\mu^a \zeta^\mu$ . Using the above definitions, we can easily find the expression for the tensor  $H_{\mu\nu}$  in terms of  $h_{\mu\nu} \equiv \tilde{h}_{\mu\nu}/M_{\text{Pl}}$ :

$$H_{\mu\nu} = \frac{\tilde{h}_{\mu\nu}}{M_{\text{Pl}}} + \frac{\partial_\mu V_\nu + \partial_\nu V_\mu}{\Lambda_3^3} - \frac{\partial_\mu V_\alpha \partial_\nu V_\alpha}{\Lambda_3^6}, \quad (13)$$

where  $V_\mu \equiv \delta_\mu^a \pi_a \Lambda_3^3$ , is a field that shifts as  $V_\mu \rightarrow V_\mu + \eta_{\mu\nu} \zeta^\nu \Lambda_3^3$  under the general coordinate transformations (the index contraction in (13) is done with  $\eta^{\mu\nu}$ ).

Then, the Lagrangian density (2) can be written in terms of the covariant tensors only. It reads as follows:

$$\mathcal{L} = M_{\text{Pl}}^2 \sqrt{g} R - \frac{M_{\text{Pl}}^2 m^2}{4} \sqrt{g} \left( H_{\mu\nu}^2 - H^2 + H_\mu^\nu H_\nu^\alpha H_\alpha^\mu - \frac{5}{4} H H_{\alpha\beta}^2 + \frac{1}{4} H^3 \right), \quad (14)$$

where all the indices are raised with the metric tensor  $g^{\mu\nu}$  (as a consequence, the signs in front of the cubic terms in (14) flip as compared with (11)). In the cubic order (14) reduces to (2) after using the gauge fixing condition  $V_\mu = 0$ .

Moreover, the external sources/fields couple to  $g_{\mu\nu}$ : For instance the Lagrangian for a scalar field  $\psi$  coupled to gravity would read as follows:

$$\mathcal{L}_\Psi = \frac{1}{2}\sqrt{g}(-g^{\mu\nu}\partial_\mu\psi\partial_\nu\psi - 2V(\psi)).$$

We will not write explicitly the couplings of  $g_{\mu\nu}$  to the external sources/field below, but will keep them in mind (see, discussions below).

To turn to the decoupling limit we decompose the vector field  $V_\mu$  as follows:

$$V_\mu = mA_\mu + \partial_\mu\pi, \quad (15)$$

where both  $A_\mu$  and  $\pi$  are kept finite in the limit (4). The field  $A_\mu$  will end up encoding the helicity  $\pm 1$  states, while the field  $\pi$  will describe the helicity-0 state.

Next we calculate the decoupling limit of the Lagrangian (14). This is done by substituting (13) and (15) into (14), and taking the limit (4). This procedure – valid for fields that decay fast enough at spatial infinity – requires some care: we introduce an infrared regulator of the theory, say a large sphere of radius  $L \gg 1/m$ , and take the radius to infinity,  $L \rightarrow \infty$ , before taking the limit (4). This hierarchy of scales enables us to put all the surface terms to zero before taking the decoupling limit.

Once the above procedure is adopted we find the following remarkable properties: (1) All the terms containing six derivatives and three helicity-0 fields, such as  $(\partial^2\pi)^3$ , that come suppressed by the scale  $\Lambda_5 \equiv (M_{\text{Pl}}m^4)^{1/5} \ll \Lambda_3$ , cancel out up to a total derivative [4]. (2) The quadratic terms in  $A$  form the Maxwell term, while all the terms that are linear in  $A$  and quadratic in  $\pi$ , such as  $\partial A \partial^2\pi \partial^2\pi$ , which would be suppressed by the scale  $\Lambda_4 \equiv (M_{\text{Pl}}m^3)^{1/4} \ll \Lambda_3$ , also cancel out up to a total derivative. (3) The only terms that survive are those suppressed by the scale  $\Lambda_3$ .

In this section we focus only on the helicity-2 and helicity-0 modes, while the terms with the helicity-1 field will be ignored until the next section, where they are shown to be harmless.

The remaining terms, after the conformal transformation  $\tilde{h}_{\mu\nu} = \bar{h}_{\mu\nu} + \eta_{\mu\nu}\pi$  that diagonalizes the quadratic action, read as follows:

$$\begin{aligned} \mathcal{L}_{\Lambda_3}^{\text{lim}} = & -\frac{1}{2}\bar{h}_{\mu\nu}\mathcal{E}^{\mu\nu\alpha\beta}\bar{h}_{\alpha\beta} + \frac{3}{2}\pi\Box\pi + \frac{3\Box\pi(\partial_\mu\pi)^2}{4\Lambda_3^3} \\ & + \frac{\bar{h}((\partial_\mu\partial_\nu\pi)^2 - (\Box\pi)^2)}{4\Lambda_3^3} + \frac{\bar{h}_{\mu\nu}(\partial_\mu\partial_\nu\pi\Box\pi - \partial_\mu\partial_\alpha\pi\partial_\nu\partial_\alpha\pi)}{2\Lambda_3^3}. \end{aligned} \quad (16)$$

Here, all the indices are raised using the flat space metric and we do not distinguish between the upper and lower cases.  $\mathcal{E}$  denotes the Einstein operator that is related to the linearized Einstein tensor  $G_{\mu\nu}$  as follows:

$$\mathcal{E}^{\mu\nu\alpha\beta}\bar{h}_{\alpha\beta} = G^{\mu\nu} = -\frac{1}{2}(\Box\bar{h}^{\mu\nu} - \partial^\mu\partial^\alpha\bar{h}_\alpha^\nu - \partial^\nu\partial^\alpha\bar{h}_\alpha^\mu + \partial^\mu\partial^\nu\bar{h} - \eta^{\mu\nu}\Box\bar{h} + \eta^{\mu\nu}\partial_\alpha\partial_\beta\bar{h}^{\alpha\beta}).$$

All the terms in (16), up to total derivatives, are invariant under the “galilean” transformations of the  $\pi$  field,  $\partial_\mu \pi \rightarrow \partial_\mu \pi + v_\mu$ , where  $v_\mu$  is some constant four-vector. Moreover, none of the cubic terms in (16) produce ghosts on any local background. Indeed, the last term in the first line of (16) is identical to the one found in the decoupling limit of the DGP model [12] (see, also [19] for related discussions). This term is known to give rise to the equations of motion that have no more than two derivatives acting on each fields [12]. To see that the rest of the terms in (16) are also safe we rewrite them as follows:

$$\bar{h} \left( (\partial_\mu \partial_\nu \pi)^2 - (\square \pi)^2 \right) = \bar{h} \left( 2\partial_0^2 \pi \Delta \pi - 2(\partial_0 \partial_j \pi)^2 - (\Delta \pi)^2 + (\partial_i \partial_j \pi)^2 \right), \quad (17)$$

and

$$\begin{aligned} \bar{h}_{\mu\nu} (\partial_\mu \partial_\nu \pi \square \pi - \partial_\mu \partial_\alpha \pi \partial_\nu \partial_\alpha \pi) &= \bar{h}_{00} (\partial_0^2 \pi \Delta \pi - (\partial_0 \partial_j \pi)^2) + \\ 2\bar{h}_{0j} (\partial_0 \partial_j \pi \Delta \pi - \partial_0 \partial_k \pi \partial_j \partial_k \pi) &+ \bar{h}_{ij} (\partial_i \partial_j \pi \square \pi - \partial_i \partial_\nu \pi \partial_j \partial_\nu \pi). \end{aligned} \quad (18)$$

These have at most two time derivatives, and will not produce any terms with more than two time derivatives in the equations of motion. Importantly, the term multiplying  $\bar{h}_{00}$  in (16) has no time derivatives, only spatial ones, showing that  $\bar{h}_{00}$  remains a Lagrange multiplier at the cubic order in the decoupling limit.

The expression (16) is invariant under the gauge transformations,  $\delta \bar{h}_{\mu\nu} = -\partial_\mu \zeta_\nu - \partial_\nu \zeta_\mu$ , due to the cancellation between the two nonlinear terms in the second line. In other words, the Bianchi identity is automatically satisfied for the tensor equation that follows from (16) by varying it w.r.t.  $\bar{h}_{\mu\nu}$ .

If we ignore coupling to external sources, one can simplify further the Lagrangian (16) by the following nonlinear transformation

$$\bar{h}_{\mu\nu} = \bar{h}'_{\mu\nu} + \frac{1}{2\Lambda_3^3} \partial_\mu \pi \partial_\nu \pi. \quad (19)$$

The resulting Lagrangian reads:

$$\mathcal{L}_{\Lambda_3}^{lim} = -\frac{1}{2} \bar{h}'_{\mu\nu} \mathcal{E}^{\mu\nu\alpha\beta} \bar{h}'_{\alpha\beta} + \frac{3}{2} \pi \square \pi + \frac{3 \square \pi (\partial_\mu \pi)^2}{4\Lambda_3^3} + \dots \quad (20)$$

As emphasized before, in the present theory it is the field  $g_{\mu\nu}$  that couples to external sources/fields. In the linearized theory the linearized source/field stress-tensor  $T_{\mu\nu}$  couples as  $\bar{h}_{\mu\nu} T_{\mu\nu}/M_{\text{Pl}}$ , which after the conformal transformation reads as follows  $(\bar{h}'_{\mu\nu} T_{\mu\nu} + \pi T)/M_{\text{Pl}}$ . These couplings, e.g., for a static source, are held fixed and finite in the decoupling limit (i.e.  $T_{\mu\nu}/M_{\text{Pl}}$  is fixed to be finite [12]). However, the diagonalization of the nonlinear terms performed by (19) would generate an additional nonlinear coupling  $\partial_\mu \pi \partial_\nu \pi T_{\mu\nu}/(M_{\text{Pl}} \Lambda_3^3)$ . Hence, to avoid complications with the additional nonlinear couplings and the modified light-cone, it is better to think of the “decoupling” limit Lagrangian (16) in which the helicity-2 and helicity-0 field mix at the cubic order. For certain sources – such as static ones – the additional



coupling  $\partial_\mu \pi \partial_\nu \pi T_{\mu\nu} / (M_{\text{Pl}} \Lambda_3^3)$  is zero, and therefore using the Lagrangian (20) in which the helicity-2 and helicity-0 are truly decoupled may be more convenient.

Interestingly, the field redefinition (19) in (16) generates in (20) the term

$$\frac{2\partial_\mu \pi \partial_\nu \pi (\partial_\mu \partial_\beta \pi \partial_\nu \partial_\beta \pi - \partial_\mu \partial_\nu \pi \square \pi) + (\partial_\mu \pi)^2 ((\square \pi)^2 - (\partial_\alpha \partial_\beta \pi)^2)}{16\Lambda_6^3}. \quad (21)$$

This is exactly the quartic Galileon introduced in [13] as a ghost-free quartic order term giving rise to two derivative equations of motion. Note that the coefficients of the cubic Galileon in (20) and the quartic Galileon in (21) are related to each other since they both originate in the cubic mixing terms of helicity-2 with helicity-0 in (16). In the present paper we ignore the quartic order terms in  $h_{\mu\nu}$ , but in general, depending on the coefficients of these terms, the quartic Galileon (21) may or may not cancel. These issues will be discussed in detail in [16], where it will be shown that in a general quartic-order theory, as soon as the cubic Galileon is present in (20), we are also bound to generate either the quartic Galileon, or a quartic mixing, or even both together accompanied by the quintic Galileon.

In practice, the quartic and higher order terms are negligible at large scales where the tensor-scalar gravity sets in [20], while they become as important as the quadratic and cubic contributions around the Vainshtein scale [21] (see also [22]). The mixing terms will be essential to address the issue whether the full theory admits superluminal propagation, as the pure Galileon terms do, (see Refs. [23, 13]).

## 4. Extracting the helicity-1 modes

The Lagrangian for the vector field in the decoupling limit is obtained by substituting (13) and (15) into (14), and taking the limit (4).

As mentioned before, the quadratic terms in  $A$  form the Maxwell term. Moreover, terms linear in  $A$  and quadratic in  $\pi$ , (*e.g.*,  $\partial A \partial^2 \pi \partial^2 \pi$ ), that would be suppressed by the scale  $\Lambda_4 \equiv (M_{\text{Pl}} m^3)^{1/4} \ll \Lambda_3$ , cancel out up to total derivatives. The terms that survive in the decoupling limit are suppressed by the scale  $\Lambda_3$ . These mix two helicity-1 fields with helicity-0,  $\partial A \partial A \partial^2 \pi$ . The resulting Lagrangian reads as follows:

$$\begin{aligned} \mathcal{L}_A = & -\frac{1}{4} F_{\mu\nu}^2 - \frac{\partial_\mu \partial_\nu \pi}{4\Lambda_3^3} (2\partial_\mu A_\alpha \partial_\nu A_\alpha + 2\partial_\alpha A_\mu \partial_\alpha A_\nu + 8\partial_\mu A_\alpha \partial_\alpha A_\nu - 12\partial_\alpha A_\alpha \partial_\mu A_\nu) \\ & - \frac{\square \pi}{4\Lambda_3^3} (-(\partial_\alpha A_\beta)^2 - 5(\partial_\mu A_\nu \partial_\nu A_\mu) + 6(\partial_\alpha A_\alpha)^2). \end{aligned} \quad (22)$$

As before, all the contractions are by  $\eta_{\mu\nu}$ , and no distinction is made between the lower and upper-case indices.

The above expression is invariant under the internal galilean transformations. However, the gauge invariance of (22) is not immediately obvious. We leave it to reader's pleasure to show that (22) reduces, up to a total derivative, to the following

Lagrangian:

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}^2 - \frac{1}{2\Lambda_3^3}A_\mu(\partial_\mu\partial_\nu - \eta_{\mu\nu}\square)(\partial_\alpha\pi F_{\alpha\nu}). \quad (23)$$

The above expression is invariant, up to a total derivative, w.r.t. the gauge transformations of the vector field,  $A_\mu \rightarrow A_\mu + \partial_\mu\chi$ , where  $\chi$  is the gauge parameter.

Furthermore, the nonlinear terms in (23) can be removed by the following nonlinear field redefinition:

$$A_\mu \rightarrow A_\mu - \frac{1}{2\Lambda_3^3}(\partial_\alpha\pi F_{\alpha\mu}). \quad (24)$$

After this transformation, and up to quartic terms, we are left with the Maxwell Lagrangian. Therefore, (22) describes helicity  $\pm 1$  modes, and does not give rise to ghosts at cubic order on any local background.

## 5. Brief comments

Since the summary of our main results has already been given in Section 1, we end this work with a few technical comments.

(i) In Section 3 we calculate the decoupling limit of (14) using the method of Ref. [6]. The method of Ref. [10], although similar to that of [6], differs from it slightly and follows more closely the Stückelberg method for the gauge fields. Furthermore, we have checked that in the approach of Ref. [10] all the six-derivative cubic terms cancel out, and four derivative ones remain. The mixing terms for helicity-2 and helicity-0 are present, and take a somewhat different form, but satisfy gauge invariance and yield automatically the Bianchi identities. One can also show that those mixing terms are reducible by a nonlinear field redefinition to the ones we obtained here, and are also removable by yet another field redefinition (if external fields/sources are ignored) at the expense of generating the higher order terms.

(ii) It is straightforward to see that the special coefficients in (2) also play a role in the construction of the Hamiltonian. Indeed, introducing the standard ADM decomposition [24] with the lapse  $N$  and shift  $N_j$  we find that the cubic order Hamiltonian of (2) is linear in  $\delta N = N - 1$ . Hence the Hamiltonian constraint is maintained in this order<sup>5</sup>. On the other hand,  $N_j$  enters quadratically and is algebraically determined, as required for massive spin-2. Note that the decoupling limit considerations do not yet guarantee positive semidefiniteness of the Hamiltonian of the full theory, and this has to be addressed separately (see discussions in [3]).

(iii) Ref. [7] showed that  $\delta N$  does get quadratic terms in the quartic order of a general massive theory (see also [5, 25]). This dependence, however, may come in a special combination with  $N_j^2$  that allows to preserve the Hamiltonian constraint and avoid the BD term, see more on this in Ref. [16].

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<sup>5</sup>Note that  $h_{00}$  enters quadratically, but this does not prevent the Hamiltonian to be linear in  $\delta N$ , which is the right variable at the nonlinear level and away from the decoupling limit.

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